Putting thinking into middle and upper secondary mathematics

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Introduction

The application of skills and knowledge to a "real world" context entails a greater set of competencies than just technical proficiency, for both the student who sits the problem and the teacher who sets it. These competencies require unpacking. A greater understanding of these competencies would enable teachers to create tasks that test what they are supposed to, with improved insight into difficulty level. Moreover, it would enable students to receive not only a better training for those tasks, but also a more meaningful education. The reflections below are an attempt to begin this unpacking process and are based on my own teaching experiences.

The current Year 12 format for problem solving assessment in Victorian education is 'extended response', which is the model that I have been adopting in mathematics classes down to Year 9. (My current and former schools are 9–12 campuses.) In this context, students have access to their Computer Algebra System (CAS) calculators, but may be asked to prove certain results along the way using their by-hand skills. When one result is required in subsequent parts, it is standard to provide the answer in the question and ask students to "show that" result using appropriate algebraic techniques.

The feedback I have recorded indicates that even technically strong students find these questions difficult. This is something of a mystery since the actual mathematics involved is usually not complex, indeed with CAS calculator technology, it often comes down to listing just the key equations or steps. This was particularly evident in my first year at Nossal High, a new select-entry government high school in the south-east of Melbourne, Victoria.

Consider the extended response problem (Figure 1) that was set for a Year 9 mid-year examination. The units of work we had covered included linear algebra, equations and relations. The question was designed to test their understanding broadly across these different aspects of intermediate linear mathematics.

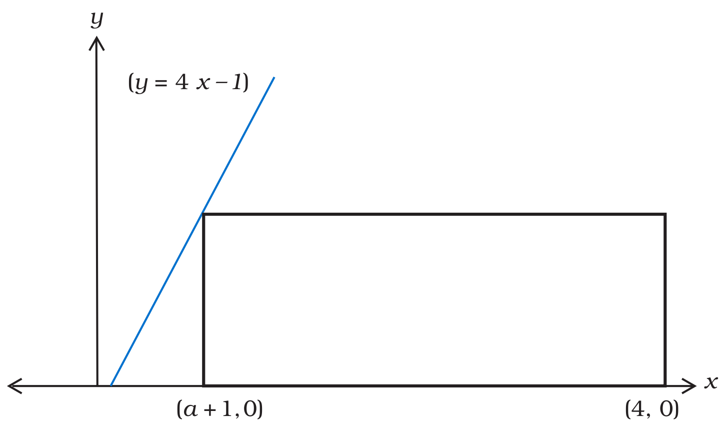


Figure 1. Year 9 mid-year examination extended response problem.

On Athletics Day, Josiah dropped a long pole onto the top corner of a rectangular platform. See the diagram on the previous page for a side view of the situation. Josiah was standing at the point (0,0) at the time.
All units are in metres.

a) Show that the height of the rectangular platform is $4a + 3$, given that the pole has the equation $y = 4x - 1$ in the above diagram.

b) Show that the perimeter of the rectangular cross-section featured above is $6a + 12$.

c) Determine how far Josiah is from the foot of the pole on the ground; assume Josiah's feet end at the point (0, 0).

d) If the *perimeter* determined in b), plus the distance along the ground from the pole to the platform equals 12.05 m, show that the value of a is –0.1. Full marks will be awarded for a correctly modelled equation leading to a solution.

e) The pole reaches 4.8 m *above the ground*. Using the gradient of the pole, find the length of platform directly underneath it.

Students at my school performed very poorly on this question, despite their high mean ability. Indeed, many of the students who showed they could appropriately substitute into formulas, accurately simplify expressions and solve linear equations, in earlier parts of the paper, had no idea how to respond to any part of this question. I had clearly underestimated its difficulty. I thought about it purely on the required technical skills for a solution, rather than the thinking skills required to draw on these techniques, or indeed even the clarity of the question itself. It begs the following questions about middle and upper secondary mathematics education.

1. How can our teaching and learning practices develop better thinkers rather than just proficient technicians? In other words, how can we help our students find engagement with mathematical concepts rather than just solution procedures?
2. How can we not only accurately gauge task difficulty, in extended response, applied contexts, when assessing aspects of the mathematics curriculum, but also test what we intend to?

Conceptual versus procedural learning

After our Year 9 students sat the above-mentioned examination, several students complained to me that the Josiah question (featured in Figure 1) seemed foreign to them. Some went so far as to claim "You didn't teach us that!" (I'm sure many readers have heard that one before!)

I believe this extreme reaction lies with the view that mathematics is about procedures rather than concepts. It seems to be borne out of student expectation that a mathematics test should canvass a range of routine problems that they have already been exposed to. (I would argue that there is certainly a place for that, but tests based exclusively on this strategy give a limited insight into a student's true understanding.) It would seem that in their previous experience (largely prior to Nossal), all tests feature 'familiar' problems, and testing for them is about recognising the pattern for which routine belongs where. Under this model of assessment, question difficulty is judged on the complexity of the routine that needs to be applied, rather than the depth of thinking required for accessing a relevant solution strategy.

At Nossal High, a great deal of care is spent developing mathematical concepts from prior understandings. However, in practice, only a portion of students engage with this introductory part of our presentations. Many students appear to only 'tune in' when worked examples are given and processes have been laid out. For such students, building understanding at the conceptual level does not appear to equate to 'success in mathematics'. To highlight this point, a Year 9 student asked me for help one day during the lesson. After a lengthy explanation covering the underpinning principles behind a solution, the student turns to me and says, "So when I see this, I do that?".

At Nossal High, academic success is highly valued by the student body. However, this can sometimes come at a cost to real learning, where flexibility, creativity and insight play, too frequently, second fiddle to 'correctness'. A cultural shift needs to occur where thinking is valued in its own right and engagement with 'rich', cross-conceptual problems are made routine.

An interesting approach to incorporating original thinking into mathematics teaching, and giving students the opportunity to create approaches for themselves, was outlined by Sullivan and colleagues (2015). Their lesson structure included posing problems that connect different aspects of mathematics together and for which students have not been shown a solution, and giving all students opportunity to engage with the key ideas. Of course, these issues mentioned above are not restricted to intermediate mathematics, but rather have a flow on effect into their Year 11 and 12 studies, where curriculum constraints become tight.

Rating difficulty levels for extended response tasks

Does a problem solving 'continuum' exist, at least in the context of Victorian upper secondary school education? That is, if we unpack all the thinking skills involved in a given question at a particular level, can we determine its difficulty level via a problem-solving scale? Does such a continuum change from topic to topic, be it how we progress along it, or the elements that constitute it? What about problem solving across several topic areas, like on exams where one question may blend together several different units of work? How does this add to problem solving complexity, if at all?

A lot of valuable work in the area of rating 'item demand' has occurred through the Programme for International Student Assessment (PISA), which, since 2000, has been testing mathematical ability in 15 year-olds across the Organisation for Economic Co-operation and Development (OECD) countries. As a means for 'objectively' assessing question difficulty, PISA has developed competency frameworks over six different areas of problem solving. (Interestingly, only one of the six entails a student's ability to work with symbols, operations and formal conventions.) Considerable effort has gone into defining and re-defining these six different areas of the framework so that cross-over is avoided and item 'raters' become more uniform in their assessment of item demand. The insights gained from PISA should help in the above mentioned project. The point of difference, however, is that, when contending with students of such diverse educational background, particularly in terms of their curricula, PISA focusses only on cognitively oriented definitions. By contrast, as a Victorian mathematics educator, it is important to know how previous experience affects students' ability to problem-solve within areas of the curriculum, as well as across them and beyond them. Nevertheless, it provides a starting point for the study.

Re-consider the Year 9 mid-year examination extended response problem presented in Figure 1. Part (a) required students to think about the 'height of the platform' in terms of the Cartesian plane on which the diagram was set. This thinking skill, captured broadly by PISA as 'representation', appeared to be beyond the capacity of many students. Should this lack of skill be attributed to their limited experience with problem solving within the Cartesian plane or is there something developmentally challenging about this kind of thinking for the average 15 year old? Perhaps in my years of experience I overlooked the conceptual gap between Josiah's tip toes being at a certain location in physical space and the mathematical idealisation of a singular point, (0, 0), at which they are purported to occur? That is, the problem is a mix of 'reality' and 'mathematical reality' and perhaps students are getting lost transitioning between the two? Furthermore, even if students could transition between the physical elements of the question and the Cartesian diagram, they still needed to realise that the height of the platform was equal to the y-value of the pole at the point of contact. And this leads to a further thinking skill that PISA labels, 'devising a strategy'.

The skill of finding the y-value of a relation, for given values of x, was well drilled. However, finding y when x is '$a + 1$' proved the stumbling block for some at this 'strategy stage', even though applying the distributive law (expanding brackets) and then collecting like terms was also well practised. This shortcoming I am suggesting lies with the student's understanding, but perhaps it lay with me in the posing of the question itself. Could the "show that" element of the question be the source of confusion rather than the thinking skills required for an answer? (Some students in higher year levels, with greater exposure to these "show that" questions, continue to struggle with this style of question. It has motivated the study of mathematical proof in our Year 11 Specialist Mathematics course.) Should I have simply asked students to "Find the height of the pole above the ground, in terms of a"? Would this have made the question more accessible?

The same concern relates to part (b), but as part (d) required the answer to part b), the use of "show that" seems more imperative. (The 2013 Mathematical Methods Exam 2 contained an extended response question with a part that really should have been expressed as a "show that". Many students were unable to solve this part and thus were denied access to the remaining 6 marks of the question. On questioning the examiner about this, his reluctance came from the point of view of assessment. It was felt that if it was expressed as a "show that", 'fudged' answers may have been hard to distinguish from genuinely derived ones. No such issues arose on the 2014 paper.)

I was genuinely surprised to find widespread struggling by students with part (c). The majority of students who sat the paper were able to determine the x-intercept of a linear function such as this one, but of those, many had no idea that it directly related to this part of the question. It would seem that the transition between 'reality' and 'mathematical reality' again proved their undoing.

With regards to part (d), students were required to subtract their part (c) answer from '$a + 1$', add it to the expression for perimeter provided in part (b), and set this equal to 12.05. On purely technical grounds, several steps were required for a solution, making it difficult, as well as the presence of fractions/decimals in the resultant equation. However, judging by the number of blank responses I witnessed for this part, it appears that again, at the strategy stage of thinking, students encountered difficulty. Nevertheless, could there have been another factor, unrelated to the mathematics or thinking skills, making this a challenging task? It could be argued that the combined length of the perimeter and the platform-pole distance is not a useful measurement, therefore, being a value lacking meaningful application, part d) becomes intangible and strange. To highlight this view, it could be claimed that the perimeter of the platform's cross-section is, by contrast, a meaningful measurement. (We may want to create some edging along this side of the platform, for instance.) However, without a connection to a justifiable purpose, perhaps the value required in part (d) was just too abstract?

And finally, part (e) was difficult because it required students to identify the gradient of the pole from the equation, apply the concept of gradient as the ratio of rise to run and appeal to information already obtained in earlier parts of the question. Devising a strategy was indeed a challenge. However, the wording of part (e) may also have been problematic. Is "the length of the platform directly underneath it" clear? I think in hindsight, by introducing a few labelled points in the diagram, and referencing them in the text of the question, I could have eliminated potential ambiguity.

What made this question so difficult? Was it the unusual blend of familiar skills required for a solution? Was it the lack of experience with Cartesian 'representation'? Or was it the manner of questioning itself? It is clear that we need our students to become better versed in problem solving strategies, as well as continue to develop their technical mastery, but we also need to ensure that the challenges are what we intend them to be, and not something else. This is where more analysis would help.

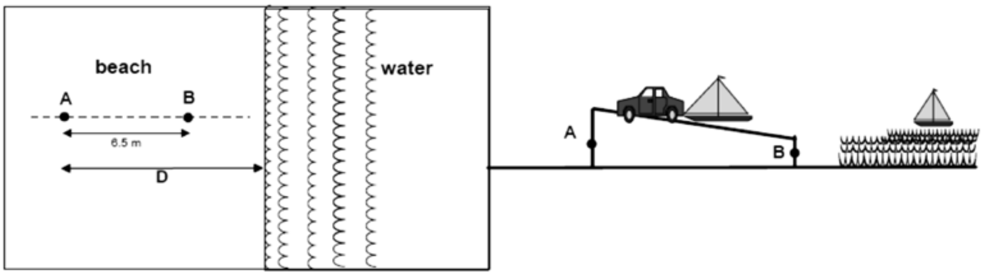
Conclusion

What is required is dedicated research. I intend to create and test a range of extended response problems, from Year 9 through to Year 12, with their thinking skills, as well as their technical skills, fully unpacked. Student feedback on these problems, as well as cross-testing with other like problems, should help inform the accuracy of the skill set they purport to test. Indeed, in cases where students are misled by the posing of the question itself, criteria can hopefully be established for eliminating these distractions. The problems created, once clarified and well-directed, need to be unit based, as well as inter-unit based. Student performances on them could be compiled and analysed, together with information on past exposure to like problems. The results of such research would lead to better assessments providing truer indicators of ability. It would also hopefully lead to better classroom practices. A well-defined conceptual framework for creating problem solving tasks in a chosen area(s) of the syllabus would enable teachers to appropriately scaffold students through problem solving lessons. Moreover, by offering these sessions regularly, with problems accessible, but challenging, at a range of depths, students would hopefully find engagement in thinking, rather than merely attaining a mastery of technical routines at the next curriculum level. The end game has to be producing better skilled and more resilient thinkers of the future. That has to be a worthwhile pursuit.

Appendix

As a final case in point, consider this Mathematics Methods Unit 3-4 problem from the 2009 trial paper produced by Insight. Several students had trouble with the question part shown, suggesting reasonable difficulty, yet the solution consisted of one simple line of arithmetic.

The diagrams below show the beach with two markers. *A* and *B* at the start and end of the boat ramp, where the base of marker *A* is 6.5 metres further up the beach than the base of marker *B*. The line *AB* is perpendicular to the water's edge.



Keith records the horizontal distance of the water's edge form the too marker *A*. He calculates that on a particular day the distance *D* metres of the water's edge from the marker *A* is a function of time *t* (in minutes from when he starts to observe the waves). [Note: A function of the form $D(t) = acos(t) + c$ relates to parts a) to d) of the question before the information below is introduced.]

Due to winds, tides and currents, on some days the waves come further up the beach and are closer together. Keith observes that on such a day the distance of the water's edge from marker *A* can be described by the equation:

$$D = (8 + S)\cos(\frac{8\pi}{3}t) + 11$$

where *S* metres is the seasonal tidal factor which varies with the factors described above. *S* is normally distributed with a mean of 2 and a standard deviation of 0.3. On a particular day the waves just reach the top of the boat ramp at marker *A*. Find the value of *S* on this day.

Figure 2: Sample question from the 2009 trial paper for Mathematics Methods Unit 3–4.

This question had lots of variables, with the above information split over two pages. In the end students had to realise that for the waves to just reach marker A, $D(t)$ must have a minimum value of zero.

PISA would classify the considerable thinking skills behind this as 'communication', for it involves deciphering a challenging amount of information spread over two pages, and 'representation'. The mathematics at that point is almost trivial. D has a minimum of zero when the amplitude of the function is equal to the (positive) mean (i.e., when $8 + S = 11$).

The difficulty level of this problem is not reducible to a competent grasp of arithmetic. Nor is it reducible to a sound knowledge of sinusoidal functions. It is thinking skills, and clearly we need to methodically factor this in if 'item demand' is to be properly evaluated.

References

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Note: Various Insight publications and materials have been referred to throughout this article.

These can be accessed via the Insight Publications website, https://www.insightpublications.com.au

The Insight publications and materials include:

- *Maths Methods CAS 2* (2009)
- *2013 Mathematical Methods Exam 2* (2013)
- *Mathematics Methods Unit 3-4 problem, Trial Examination paper* (2009).